Theorem:-1 The linear span $L(S)$ of a subset S of a vector space $V(F)$ is the smallest subspace of $V(F)$ containing S i.e. $L(S)=\{S\}$

Proof:- let $\alpha, \beta \in L(S)$

$$\Rightarrow \quad \alpha = \sum_{i=1}^{n} a_i \alpha_i \quad \& \quad \beta = \sum_{j=1}^{m} b_j \beta_j \quad \text{—— (i)}$$

where $a_i \in F, \alpha_i \in S \; (i=1,2,\dots n)$

& $b_j \in F, \beta_j \in S \; (j=1,2,\dots m)$

If a & b are two arbitary element of F,

then $$a\alpha + b\beta = a\left(\sum_{i=1}^{n} a_i \alpha_i\right) + b\left(\sum_{j=1}^{m} b_j \beta_j\right) \quad (\text{by (i)})$$

$$= \sum_{i=1}^{n} (a a_i)\alpha_i + \sum_{j=1}^{m} (b b_j)\beta_j$$

$\left\{ \begin{array}{l} \text{since } a, a_i \in F \Rightarrow a a_i \in F \; (i=1,2,\dots n) \\ \& \; b, b_j \in F \Rightarrow b b_j \in F \; (j=1,2,\dots m) \end{array} \right\}$
By the defn. of field

$\Rightarrow$ $a\alpha + b\beta$ is a LC of finite elements $\{\alpha_1, \alpha_2, \dots \alpha_n, \beta_1, \beta_2, \dots \beta_m\}$ of S.

$\Rightarrow \quad a\alpha + b\beta \in L(S)$

Hence $a, b \in F$ & $\alpha, \beta \in L(S) \Rightarrow a\alpha + b\beta \in L(S)$

$\Rightarrow$ $L(S)$ is subspace of vector space $V(F)$

Now for every element α of S,

$\alpha = 1 \cdot \alpha \Rightarrow \alpha$ is L.C. of element of S

$\Rightarrow \alpha \in L(S)$

$\Rightarrow S \subset L(S)$

let W be a subspace of V(F) ~~which~~ s.t.

$S \subset W$

since W is closed w.r. to vector addition and scalar multiplication.

$\Rightarrow L(S) \subset W$

$\Rightarrow L(S) = \{S\}$

$\Rightarrow$ L(S) is the smallest subspace of V(F) containning S.

Linear sum of two subspaces:-

Let W_1 & W_2 are two subspace of vector space V(F) then their linear sum $W_1 + W_2$ is defined as

$$W_1 + W_2 = \{\alpha_1 + \alpha_2 \mid \alpha_1 \in W_1, \alpha_2 \in W_2\}$$

since $0 \in W_1$ & $0 \in W_2$

$\alpha_1 \in W_1, 0 \in W_2 \Rightarrow \alpha_1 + 0 = \alpha_1 \in W_1 + W_2$

$\Rightarrow W_1 \subset W_1 + W_2$

similarly $0 \in W_2, \alpha_2 \in W_2 \Rightarrow 0 + \alpha_2 = \alpha_2 \in W_1 + W_2$

$\Rightarrow W_2 \subset W_1 + W_2$

Hence the linear sum of two subspace contains both the subspace.

Theorem :- 2 The linear sum of two subspace of a vector space is subspace.

Proof:→ let K and W be two subspace of vector space V(F). linear sum of K & W

$$K+W = \{ k_1 + w_1 \mid k_1 \in K, w_1 \in W \}$$

let $\alpha, \beta \in K+W$

$\Rightarrow \alpha = k_1 + w_1, \quad \beta = k_2 + w_2$ —— (i)

where $k_1, k_2 \in K$, $w_1, w_2 \in W$

Since K & W are two subspace so for

$a, b \in F$ & $k_1, k_2 \in K \Rightarrow ak_1 + bk_2 \in K$ —(ii)

$a, b \in F$ & $w_1, w_2 \in W \Rightarrow aw_1 + bw_2 \in W$ —(iii)

from (ii) & (iii)

$ak_1 + bk_2 \in K$, $aw_1 + bw_2 \in W$

$$(ak_1 + bk_2) + (aw_1 + bw_2) \in K+W$$

$$\Rightarrow (ak_1 + aw_1) + (bk_2 + bw_2) \in K+W$$

$$\Rightarrow a(k_1 + w_1) + b(k_2 + w_2) \in K+W$$

$$\Rightarrow a\alpha + b\beta \in K+W \quad \text{(by (i))}$$

so for $a, b \in F$; $\alpha, \beta \in K+W$

$\Rightarrow a\alpha + b\beta \in K+W$

$\Rightarrow$ K+W is subspace of vector space V(F).

Theorem: 3 If K and W are two subspaces of a vector space $V(F)$, then

$$K+W = L\{K \cup W\} = \{K \cup W\}$$

Proof:- Since K and W are two subspaces of $V(F) \Rightarrow 0 \in K, 0 \in W$

$\Rightarrow K \neq \phi, W \neq \phi$

Let $k_1 \in K \Rightarrow k_1 = k_1 + 0 \in K+W$ ($\because 0 \in W$)

$\Rightarrow K \subseteq K+W$ —— (i)

Similarly $W \subseteq K+W$ —— (ii)

from (i) & (ii)

$K \cup W \subseteq K+W$ —— (iii)

By previous theorem $K+W$ is also a subspace of $V(F)$.

so from (iii), $K+W$ is a subspace of $V(F)$ s.t. it contains $K \cup W$.

let $\alpha = k_1 + w_1 \in K+W$, where $k_1 \in K$; $w_1 \in W_1$

$\Rightarrow k_1, w_1 \in K \cup W$

$\because k_1 + w_1 = 1 \cdot k_1 + 1 \cdot w_1$ where $1 \in F$

$\Rightarrow$ Vector $\alpha = k_1 + w_1$ can be expressed as a LC of elements $k_1, w_1 \in K \cup W$

Hence $k_1 + w_1 \in L\{K \cup W\}$

$\Rightarrow K+W \subseteq L\{K \cup W\}$ —— (iv)

{ Since the linear span $L(S)$ of subset S of a vector space $V(F)$ is the smallest subspace of $V(F)$ containning S i.e. $L(S) = \{S\}$ (36)

$S \to K \cup W$

since $L\{K \cup W\}$ is the smallest subspace of $V(F)$ containning $K \cup W$. &

from (iii) $K \cup W \subseteq K + W$

$\Rightarrow L(K \cup W) \subseteq K + W$ — (v)

so from (iv) & (v)

$L(K \cup W) = K + W = \{K \cup W\}$